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Some remarks on the Lax pairs for a one-dimensional small-polaron model and the one-dimensional Hubbard model

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Abstract. Ambiguities in the Lax pairs for a one-dimensional small-polaron model and the one-dimensional Hubbard model are discussed.

1. Introduction

The one-dimensional (1D) small-polaron model describing the motion of an additional electron in a polaronic crystal is given by the Hamiltonian

$$\mathscr{H} = W\sum_{j=1}^{N} n_j - J\sum_{j=1}^{N}(a_j^\dagger a_{j-1} + a_{j-1}^\dagger a_j) + V\sum_{j=1}^{N} n_j n_{j-1}. \tag{1}$$

Here $a_j^\dagger$ and a_j are, respectively, creation and annihilation operators at lattice site j in a one-dimensional chain of N sites, and satisfy the usual anticommutation relations

$$\{a_i, a_j\} = \{a_i^\dagger, a_j^\dagger\} = 0 \qquad \{a_i, a_j^\dagger\} = \delta_{ij} \tag{2}$$

and n_j is the density operator

$$n_j = a_j^\dagger a_j \tag{3}$$

while the concrete expressions for W, J and V can be found in the paper of Fedyanin and Yushankay [1]. This model was first studied by Pu and Zhao [2] and then by Zhou *et al* [3] in the framework of the quantum inverse scattering method (QISM) [4-6].

Another interesting completely integrable system in condensed matter theory is the well known 1D Hubbard model

$$\mathscr{H} = -\sum_{j,s}(a_{js}^\dagger a_{j-1s} + a_{j-1s}^\dagger a_{js}) + U\sum_j (n_{j\uparrow} - \tfrac{1}{2})(n_{j\downarrow} - \tfrac{1}{2}) + \mu\sum_{j,s}(n_{js} - \tfrac{1}{2}). \tag{4}$$

Here U is the coupling constant describing the Coulomb interaction and μ is the chemical potential. s represents the two components of the fermions ($s = \uparrow$ or $\downarrow$). As usual, a_{ir} and $a_{js}^\dagger$ satisfy

$$\{a_{ir}, a_{js}\} = \{a_{ir}^\dagger, a_{js}^\dagger\} = 0 \qquad \{a_{ir}, a_{js}^\dagger\} = \delta_{ij}\delta_{rs}. \tag{5}$$

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The model (4) has been shown [7, 8] to possess an infinite number of conservation laws by identifying a two-dimensional (2D) lattice statistical model for which a one-parameter family of transfer matrices commutes with the Hamiltonian. The Lax pair as well as the solution to the Yang-Baxter relations for the Hamiltonian (4) with the chemical potential term vanishing has also been obtained [9-11].

In this paper we present two different forms of the Lax pairs for a 1D small-polaron model and the 1D Hubbard models. We find that not all forms of the Lax pairs are physically reasonable when we attempt to tackle the problems in the framework of QISM.

2. 1D small-polaron model

Applying the well known Jordan-Wigner transformation for a_j, $a_j^\dagger$ and n_j

$$a_j = \exp\left(\mathrm{i}\pi \sum_{l=1}^{j-1} \sigma_l^+ \sigma_l^-\right)\sigma_j^- \qquad a_j^\dagger = \exp\left(\mathrm{i}\pi \sum_{l=1}^{j-1} \sigma_l^+ \sigma_l^-\right)\sigma_j^+ \qquad n_j = \frac{1+\sigma_j^z}{2} \tag{6}$$

with $\sigma_j^\pm = \frac{1}{2}(\sigma_j^x \pm \mathrm{i}\sigma_j^y)$ and σ_j^x, σ_j^y, σ_j^z being Pauli spin operators at lattice site j, we have

$$H = -\sum_{j=1}^{N}\left(J(\sigma_j^+\sigma_{j-1}^- + \sigma_j^-\sigma_{j-1}^+) - \frac{V}{4}\sigma_j^z\sigma_{j-1}^z\right) + \frac{W+V}{2}\sum_{j=1}^{N}\sigma_j^z + \frac{N}{2}\left(W + \frac{V}{2}\right). \tag{7}$$

Here the periodic boundary condition is imposed. Thus, the problem reduces to the study of the Heisenberg XXZ model in an external magnetic field parallel to the z direction. The equations of motion are

$$\begin{aligned}
\dot{\sigma}_j^+ &= \mathrm{i}[J(\sigma_{j+1}^+ + \sigma_{j-1}^+)\sigma_j^z - \tfrac{1}{2}V\sigma_j^+(\sigma_{j+1}^z + \sigma_{j-1}^z) + (W+V)\sigma_j^+] \\
\dot{\sigma}_j^- &= -\mathrm{i}[J(\sigma_{j+1}^- + \sigma_{j-1}^-)\sigma_j^z - \tfrac{1}{2}V\sigma_j^-(\sigma_{j+1}^z + \sigma_{j-1}^z) + (W+V)\sigma_j^-] \\
\dot{\sigma}_j^z &= 2\mathrm{i}J[\sigma_j^+(\sigma_{j+1}^- + \sigma_{j-1}^-) - (\sigma_{j+1}^+ + \sigma_{j-1}^+)\sigma_j^-].
\end{aligned} \tag{8}$$

In QISM, instead of directly considering the equations of motion, it turns out to be more fruitful to study an operator version of auxiliary problem

$$\phi_{j+1} = L_j\phi_j \qquad \dot{\phi}_j = M_j\phi_j. \tag{9}$$

The consistency condition yields the Lax equation

$$\dot{L}_j = M_{j+1}L_j - L_jM_j. \tag{10}$$

This implies that a transfer matrix

$$\tau_N = \mathrm{tr}(L_N L_{N-1} \ldots L_1) \tag{11}$$

does not depend on time under the periodic boundary condition. Thus, the corresponding system possesses an infinite number of conservation laws, which in turn are related to the integrability of the system. In our case, it is easy to check that the equations of motion (8) are equivalent to the Lax equation (10) with

$$L_j = \begin{pmatrix} \dfrac{a+b}{2} + \dfrac{a-b}{2}\sigma_j^z & c\sigma_j^- \\ c\sigma_j^+ & \dfrac{a+b}{2} - \dfrac{a-b}{2}\sigma_j^z \end{pmatrix} \tag{12}$$

and

$$M_j = \begin{pmatrix} f\sigma_j^+\sigma_{j-1}^- + g\sigma_j^-\sigma_{j-1}^+ - d\sigma_j^z\sigma_{j-1}^z + d(\sigma_j^z + \sigma_{j-1}^z) + d_0 & p(\sigma_j^-\sigma_{j-1}^z - \sigma_j^z\sigma_{j-1}^-) + q(\sigma_j^- + \sigma_{j-1}^-) \\ -p(\sigma_j^+\sigma_{j-1}^z - \sigma_j^z\sigma_{j-1}^+) + q(\sigma_j^+ + \sigma_{j-1}^+) & g\sigma_j^+\sigma_{j-1}^- + f\sigma_j^-\sigma_{j-1}^+ - d\sigma_j^z\sigma_{j-1}^z - d(\sigma_j^z + \sigma_{j-1}^z) - d_0 \end{pmatrix}. \tag{13}$$

Here

$$\begin{aligned} &f = -\mathrm{i}J\frac{a-b}{a} && g = \mathrm{i}J\frac{(a-b)b+c^2}{b^2-c^2} && d = \mathrm{i}\frac{V}{4}\frac{c^2}{b^2-c^2} \\ &d_0 = -\mathrm{i}\frac{W+V}{2} && p = \mathrm{i}\frac{V}{2}\frac{bc}{b^2-c^2} && q = -\mathrm{i}\frac{J}{2}\frac{(a^2-b^2+c^2)c}{a(b^2-c^2)} \end{aligned} \tag{14}$$

and a, b, and c are given by the usual Baxter parametrisation [12]

$$a : b : c = \sin(\lambda + \eta) : \sin(\lambda - \eta) : \sin 2\eta. \tag{15}$$

However, the transfer matrix thus constructed does not lead to the same energy eigenvalue as that obtained by using the coordinate Bethe ansatz method. This is why we want to search for another form of Lax pair for the model. We have recently shown that this deficiency can be removed by identifying a special 2D statistical model for which a one-parameter family of transfer matrices also commutes with (7) [3]. Explicitly, L_j is given by

$$L_j = \begin{pmatrix} \frac{a_+ + b_+}{2} + \frac{a_+ - b_+}{2}\sigma_j^z & c\sigma_j^- \\ c\sigma_j^+ & \frac{a_- + b_-}{2} - \frac{a_- - b_-}{2}\sigma_j^z \end{pmatrix}. \tag{16}$$

Accordingly, (10) may be viewed as a matrix difference equation for the unknown matrices M_j for $j = 1, 2, \ldots, N$ if the time derivatives of the spin operators are replaced by the equations of motion (8).

After a tedious but straightforward calculation, we obtain explicitly the form of M_j as follows:

$$M_j = \begin{pmatrix} f_+\sigma_j^+\sigma_{j-1}^- + g_-\sigma_j^-\sigma_{j-1}^+ - d\sigma_j^z\sigma_{j-1}^z + d(\sigma_j^z + \sigma_{j-1}^z) + d_0 & p_-\sigma_j^-\sigma_{j-1}^z - p_+\sigma_j^z\sigma_{j-1}^- + q_-\sigma_j^- + q_+\sigma_{j-1}^- \\ -p_+\sigma_j^+\sigma_{j-1}^z + p_-\sigma_j^z\sigma_{j-1}^+ + q_+\sigma_j^+ + q_-\sigma_{j-1}^+ & g_+\sigma_j^+\sigma_{j-1}^- + f_-\sigma_j^-\sigma_{j-1}^+ - d\sigma_j^z\sigma_{j-1}^z - d(\sigma_j^z + \sigma_{j-1}^z) - d_0 \end{pmatrix}. \tag{17}$$

The constants appearing in (17) are given by

$$\begin{aligned} &f_\pm = -\mathrm{i}J\frac{a_\pm - b_\pm}{a_\pm} \qquad g_\pm = \mathrm{i}J\frac{(a_\mp - b_\mp)b_\pm + c^2}{b_+b_- - c^2} \\ &d = \mathrm{i}\frac{V}{4}\frac{c^2}{b_+b_- - c^2} \qquad d_0 = -\mathrm{i}\frac{W+V}{2} \\ &p_\pm = \mathrm{i}\frac{V}{2}\frac{b_\pm c}{b_+b_- - c^2} = -\mathrm{i}\frac{J}{2}\frac{(a_+a_- + b_+b_- - c^2)c}{a_\pm(b_+b_- - c^2)} \\ &q_\pm = -\mathrm{i}\frac{J}{2}\frac{(a_+a_- - b_+b_- + c^2)c}{a_\pm(b_+b_- - c^2)}. \end{aligned} \tag{18}$$

From (18) we have

$$a_+ b_+ = a_- b_- \tag{19}$$

and

$$-\frac{V}{2J} = \frac{a_+ a_- + b_+ b_- - c^2}{a_+ b_+ + a_- b_-}. \tag{20}$$

Now note that the coupling constants J and V are independent of the spectral parameter λ. Thus, we have rederived the results presented in our previous work [3]. A natural parametrisation for (20) and (21) is

$$a_+ : b_+ : a_- : b_- : c = \xi \sin(\lambda + \eta) : \xi^{-1} \sin(\lambda - \eta) : \xi^{-1} \sin(\lambda + \eta) : \xi \sin(\lambda - \eta) : \sin 2\eta \tag{21}$$

with

$$\xi = \sec \alpha \cos(\lambda - \eta + \alpha) \sec(\lambda - \eta). \tag{22}$$

By this parametrisation, J, V and W are given by

$$J : -\frac{V}{2} : (W + V) = 1 : \cos 2\eta : 2 \sin 2\eta \tan \alpha. \tag{23}$$

Let us now transform back into the original fermion operators. For this purpose, we introduce a gauge transformation

$$\mathscr{L}_j = V_{j+1} L_j V_j^{-1} \tag{24}$$

with

$$V_j = \begin{pmatrix} \exp\left(\frac{i\pi}{2} \sum_{l=1}^{j-1} n_l\right) & 0 \\ 0 & \exp\left(-\frac{i\pi}{2} \sum_{l=1}^{j-1} n_l\right) \end{pmatrix}. \tag{25}$$

It is easy to see that the counterpart of the Lax pair, M_j, transforms as

$$\mathscr{M}_j = V_j M_j V_j^{-1} + \dot{V}_j V_j^{-1}. \tag{26}$$

Substituting (16) and (17) into (24) and (26), we immediately obtain

$$\mathscr{L}_j = \begin{pmatrix} b_+ - (b_+ - ia_+) n_j & ca_j \\ -ica_j^\dagger & a_- - (a_- + ib_-) n_j \end{pmatrix} \tag{27}$$

and

$$\mathscr{M}_j = \begin{pmatrix} [if_+ + (i-1)J] a_j^\dagger a_{j-1} + [-ig_- + (i+1)J] a_{j-1}^\dagger a_j - 4d n_j n_{j-1} + 4d(n_j + n_{j-1}) - 3d + d_0 & 2p_- n_{j-1} a_j - 2ip_+ n_j a_{j-1} + (q_- - p_-) a_j + i(q_+ + p_+) a_{j-1} \\ -2p_+ a_j^\dagger n_{j-1} - 2ip_- a_{j-1}^\dagger n_j + (q_+ + p_+) a_j^\dagger - i(q_- - p_-) a_{j-1}^\dagger & [-ig_+ + (i+1)J] a_j^\dagger a_{j-1} + [if_- + (i-1)J] a_{j-1}^\dagger a_j - 4d n_i n_{j-1} + d - d_0 \end{pmatrix}. \tag{28}$$

Correspondingly, we can also construct another form of the Lax pair in fermions from (12) and (13)

$$\mathscr{L}_j = \begin{pmatrix} b - (b - ia) n_j & ca_j \\ -ica_j^\dagger & a - (a + ib) n_j \end{pmatrix} \tag{29}$$

and

$$\mathcal{M}_j = \begin{pmatrix} [\mathrm{i}f + (\mathrm{i}-1)J]a_j^\dagger a_{j-1} + [-\mathrm{i}g + (\mathrm{i}+1)J]a_{j-1}^\dagger a_j - 4dn_jn_{j-1} + 4d(n_j + n_{j-1}) - 3d + d_0 & 2p(n_{j-1}a_j - \mathrm{i}n_ja_{j-1}) + (q-p)a_j + \mathrm{i}(q+p)a_{j-1} \\ -2p(a_j^\dagger n_{j-1} + \mathrm{i}a_{j-1}^\dagger n_j) + (q+p)a_j^\dagger - \mathrm{i}(q-p)a_{j-1}^\dagger & [-\mathrm{i}g + (\mathrm{i}+1)J]a_j^\dagger a_{j-1} + [\mathrm{i}f + (\mathrm{i}-1)J]a_{j-1}^\dagger a_j - 4dn_jn_{j-1} + d - d_0 \end{pmatrix}. \tag{30}$$

Thus we have completed the representation of the Lax pairs for a 1D small-polaron model.

3. 1D Hubbard model

The Hamiltonian (4) can be brought into the form

$$H = -\sum_{j=1}^{N} (\sigma_j^+\sigma_{j-1}^- + \sigma_j^-\sigma_{j-1}^+ + \tau_j^+\tau_{j-1}^- + \tau_j^-\tau_{j-1}^+) + \frac{U}{4}\sum_{j=1}^{N} \sigma_j^z\tau_j^z + \frac{\mu}{2}\sum_{j=1}^{N} (\sigma_j^z + \tau_j^z) \tag{31}$$

using the Jordan-Wigner transformation

$$\begin{aligned} a_{j\uparrow} &= \exp\left(\mathrm{i}\pi \sum_{l=1}^{j-1} \sigma_l^+\sigma_l^-\right)\sigma_j^- \qquad a_{j\uparrow}^\dagger = \exp\left(\mathrm{i}\pi \sum_{l=1}^{j-1} \sigma_l^+\sigma_l^-\right)\sigma_l^+ \qquad n_{j\uparrow} = \frac{1+\sigma_j^z}{2} \\ a_{j\downarrow} &= \exp\left(\mathrm{i}\pi \sum_{l=1}^{N} \sigma_l^+\sigma_l^-\right)\exp\left(\mathrm{i}\pi \sum_{l=1}^{j-1} \tau_l^+\tau_l^-\right)\tau_j^- \\ a_{j\downarrow}^\dagger &= \exp\left(\mathrm{i}\pi \sum_{l=1}^{N} \sigma_l^+\sigma_l^-\right)\exp\left(\mathrm{i}\pi \sum_{l=1}^{j-1} \tau_l^+\tau_l^-\right)\tau_j^+ \qquad n_{j\downarrow} = \frac{1+\tau_j^z}{2}. \end{aligned} \tag{32}$$

Here the sum over j is from 1 to N, and the periodic boundary condition is imposed. From (31) we see that our problem reduces to that of a pair of XY models with the presence of magnetic fields coupled to each other. Then, the equations of motion are as follows:

$$\begin{aligned} \dot{\sigma}_j^+ &= \mathrm{i}[(\sigma_{j+1}^+ + \sigma_{j-1}^+)\sigma_j^z + \tfrac{1}{2}U\sigma_j^+\tau_j^z + \mu\sigma_j^+] \\ \dot{\sigma}_j^- &= -\mathrm{i}[(\sigma_{j+1}^- + \sigma_{j-1}^-)\sigma_j^z + \tfrac{1}{2}U\tau_j^z\sigma_j^- + \mu\sigma_j^-] \\ \dot{\sigma}_j^z &= 2\mathrm{i}[\sigma_j^+(\sigma_{j+1}^- + \sigma_{j-1}^-) - (\sigma_{j+1}^+ + \sigma_{j-1}^+)\sigma_j^-] \end{aligned} \tag{33}$$

with similar equations for τ spins.

As in the case of a 1D small-polaron model, we can also construct two different forms of the Lax pairs for this model. One is

$$L_j = I_0 L_j^{(\sigma)} L_j^{(\tau)} I_0 \tag{34}$$

$$\begin{aligned} M_j = I_0^{-1}(M_j^{(\sigma)} + M_j^{(\tau)})I_0 &+ 2\mathrm{i}\frac{c}{a}\sinh h[(\sigma_j^+\sigma_0^- - \sigma_j^-\sigma_0^+)\tau_0^z + \sigma_0^z(\tau_j^+\tau_0^- - \tau_j^-\tau_0^+)] \\ &- \mathrm{i}\frac{U}{4}\left(1 + \frac{2b^2}{c^2}\right)\sigma_0^z\tau_0^z \end{aligned} \tag{35}$$

with

$$I_0 = \cosh\frac{h}{2} + \sigma_0^z\tau_0^z \sinh\frac{h}{2} \tag{36}$$

and

$$U = \frac{2c^2}{ab} \sinh 2h. \tag{37}$$

Here, $L_j^{(\sigma)}$, $L_j^{(\tau)}$, $M_j^{(\sigma)}$ and $M_j^{(\tau)}$ are defined by setting

$$\eta = \pi/4 \qquad J = 1 \qquad V = 0 \qquad W = \mu \tag{38}$$

in (12) and (13). In (34), and in subsequent equations, we have used Pauli spin matrices σ_0 and τ_0.

The other form of the Lax pairs for the 1D Hubbard model can be constructed in the following way. Let us first consider the case $U = 0$. Then, the corresponding equations of motion can be cast into the form

$$\dot{L}_j^{(\sigma)} = M_{j+1}^{(\sigma)} L_j^{(\sigma)} - L_j^{(\sigma)} M_j^{(\sigma)} \tag{39}$$

$$\dot{L}_j^{(\tau)} = M_{j+1}^{(\tau)} L_j^{(\tau)} - L_j^{(\tau)} M_j^{(\tau)}. \tag{40}$$

Here, $L_j^{(\sigma)}$, $L_j^{(\tau)}$, $M_j^{(\sigma)}$ and $M_j^{(\tau)}$ are defined by setting (38) in (16) and (17). Using this result, we can combine the equations of motion (33) for the case $U \neq 0$ into

$$\frac{d\tilde{L}_j}{dt} = \tilde{M}_{j+1}\tilde{L}_j - \tilde{L}_j\tilde{M}_j - i\frac{U}{4}[\tilde{L}_j, \sigma_j^z \tau_j^z] \tag{41}$$

where

$$\tilde{L}_j = L_j^{(\sigma)} L_j^{(\tau)} \tag{42}$$

$$\tilde{M}_j = M_j^{(\sigma)} + M_j^{(\tau)}. \tag{43}$$

The remaining question is how to put (41) into the form of Lax equation (10). Guided by the known results previously found in [8], we choose

$$L_j = I_0 \tilde{L}_j I_0. \tag{44}$$

Thus, (41) can be rewritten as

$$\dot{L}_j = I_0 \tilde{M}_{j+1} I_0^{-1} L_j - L_j I_0^{-1} \tilde{M}_j I_0 - i(U/4)[L_j, \sigma_j^z \tau_j^z]. \tag{45}$$

Now note that

$$I_0 \tilde{M}_j I_0^{-1} = I_0^{-1} \tilde{M}_j I_0 + Q_j + Q_{j-1}^{\dagger} \tag{46}$$

with

$$Q_j = 2ic \sinh h \left[\left(\frac{\sigma_j^+ \sigma_0^-}{a_+} - \frac{\sigma_j^- \sigma_0^+}{a_-} \right) \tau_0^z + \sigma_0^z \left(\frac{\tau_j^+ \tau_0^-}{a_+} - \frac{\tau_j^- \tau_0^+}{a_-} \right) \right]. \tag{47}$$

Then, (45) becomes

$$\dot{L}_j = (I_0^{-1} \tilde{M}_{j+1} I_0 + Q_{j+1}) L_j - L_j (I_0^{-1} \tilde{M}_j I_0 + Q_j) + Q_j^{\dagger} L_j + L_j Q_j - i(U/4)[L_j, \sigma_j^z \tau_j^z]. \tag{48}$$

A lengthy but simple algebraic calculation shows that (48) may be expressed in the Lax form (10) with

$$M_j = I_0^{-1} \tilde{M}_j I_0 + Q_j - i\frac{U}{4}\left(1 + \frac{2b_+ b_-}{c^2}\right) \sigma_0^z \tau_0^z \tag{49}$$

for a proper choice of the coupling parameters, i.e.

$$U = \frac{2c^2}{a_+ b_+} \sinh 2h = \frac{2c^2}{a_- b_-} \sinh 2h. \tag{50}$$

This is consistent with the result of Zhou and Tang [8].

Transforming back to the original fermion operators, we immediately obtain explicit results on the Lax pairs for the 1D Hubbard model (4). Explicitly, one is

$$\mathscr{L}_j = \begin{pmatrix} -e^{-h} k_{j\uparrow} k_{j\downarrow} & -k_{j\uparrow} a_{j\downarrow} & i a_{j\uparrow} k_{j\downarrow} & i\, e^{h} a_{j\uparrow} a_{j\downarrow} \\ -i k_{j\uparrow} a_{j\downarrow}^{\dagger} & e^{-h} k_{j\uparrow} l_{j\downarrow} & e^{-h} a_{j\uparrow} a_{j\downarrow}^{\dagger} & i a_{j\uparrow} l_{j\downarrow} \\ a_{j\uparrow}^{\dagger} k_{j\downarrow} & e^{-h} a_{j\uparrow}^{\dagger} a_{j\downarrow} & e^{-h} l_{j\uparrow} k_{j\downarrow} & l_{j\uparrow} a_{j\downarrow} \\ -i\, e^{h} a_{j\uparrow}^{\dagger} a_{j\downarrow}^{\dagger} & a_{j\uparrow}^{\dagger} l_{j\downarrow} & i l_{j\uparrow} a_{j\downarrow}^{\dagger} & -e^{h} l_{j\uparrow} l_{j\downarrow} \end{pmatrix} \tag{51}$$

with

$$k_{js} = b - (b - ia) n_{js} \qquad l_{js} = a - (a + ib) n_{js} \tag{52}$$

and

$$\mathscr{M}_j = \begin{pmatrix} \delta - i\mu + m_j & i\chi_{j\downarrow}(h) & \chi_{j\uparrow}(h) & 0 \\ i\chi_{j\downarrow}^{\dagger}(-h) & -\delta + m_j & 0 & -\chi_{j\uparrow}(-h) \\ -\chi_{j\uparrow}^{\dagger}(-h) & 0 & -\delta + m_j & i\chi_{j\downarrow}(-h) \\ 0 & \chi_{j\uparrow}^{\dagger}(h) & i\chi_{j\downarrow}^{\dagger}(h) & \delta + i\mu + m_j \end{pmatrix} \tag{53}$$

with

$$\delta = i\frac{U}{4}\left(1 + \frac{2b^2}{c^2}\right) \qquad m_j = \left(i - \frac{b}{a}\right) \sum_{s=\uparrow,\downarrow} (a_{js}^{\dagger} a_{j-1s} + a_{j-1s}^{\dagger} a_{js})$$
$$\chi_{js}(h) = \frac{c}{a} (e^{h} a_{js} - i\, e^{-h} a_{j-1s}). \tag{54}$$

The other pair is

$$\mathscr{L}_j = \begin{pmatrix} -e^{-h} k_{j\uparrow} k_{j\uparrow} & -k_{j\uparrow} a_{j\downarrow} & i a_{j\uparrow} k_{j\downarrow} & i\, e^{h} a_{j\uparrow} a_{j\downarrow} \\ -i k_{j\uparrow} a_{j\downarrow}^{\dagger} & e^{-h} k_{j\uparrow} l_{j\downarrow} & e^{-h} a_{j\uparrow} a_{j\downarrow}^{\dagger} & i a_{j\uparrow} l_{j\downarrow} \\ a_{j\uparrow}^{\dagger} k_{j\downarrow} & e^{-h} a_{j\uparrow}^{\dagger} a_{j\downarrow} & e^{-h} l_{j\uparrow} k_{j\uparrow} & l_{j\uparrow} a_{j\downarrow} \\ -i\, e^{h} a_{j\uparrow}^{\dagger} a_{j\downarrow}^{\dagger} & a_{j\uparrow}^{\dagger} l_{j\downarrow} & i l_{j\uparrow} a_{j\downarrow}^{\dagger} & -e^{h} l_{j\uparrow} l_{j\downarrow} \end{pmatrix} \tag{55}$$

with

$$k_{js} = b_+ - (b_+ - ia_+) n_{js} \qquad l_{js} = a_- - (a_- + ib_-) n_{js} \tag{56}$$

and

$$\mathscr{M}_j = \begin{pmatrix} \delta - i\mu + m_j & i\chi_{j\downarrow -}(h) & \chi_{j\uparrow -}(h) & 0 \\ i\chi_{j\downarrow +}^{\dagger}(-h) & -\delta + m_j & 0 & -\chi_{j\uparrow -}(-h) \\ -\chi_{j\uparrow +}^{\dagger}(-h) & 0 & -\delta + m_j & i\chi_{j\downarrow -}(-h) \\ 0 & \chi_{j\uparrow +}^{\dagger}(h) & i\chi_{j\downarrow +}^{\dagger}(h) & \delta + i\mu + m_j \end{pmatrix} \tag{57}$$

with

$$\delta = i\frac{U}{4}\left(1 + \frac{2b_+ b_-}{c}\right) \qquad m_j = \sum_{s=\uparrow,\downarrow} \left[\left(i - \frac{b_+}{a_+}\right) a_{js}^{\dagger} a_{j-1s} + \left(i - \frac{b_-}{a_-}\right) a_{j-1s}^{\dagger} a_{js}\right]$$
$$\chi_{js\pm}(h) = c\left(\frac{e^{h}}{a_{\pm}} a_{js} - i\frac{e^{-h}}{a_{\mp}} a_{j-1s}\right). \tag{58}$$

This completes our analysis for the 1D Hubbard model.

4. Conserved currents

As is well known, a completely integrable model exhibits an infinite number of conserved currents commuting with each other. The generating functional for those conserved currents is the row-to-row transfer matrix [13]. For later use, let us introduce an auxiliary space variable o. Then, the transfer matrix may be written as

$$\tau_N(\lambda)=\mathrm{tr}_o[L_{No}(\lambda)L_{N-1o}(\lambda)\ldots L_{1o}(\lambda)] \tag{59}$$

with the local monodromy matrix $L_{jo}(\lambda)$ acting in the tensor product of the space V_j and the auxiliary space V_o, and the trace as well as the matrix products are carried out in the auxiliary space V_o.

A local monodromy matrix is called regular if it satisfies the condition

$$L_{jo}(\eta)=P_{jo} \tag{60}$$

where η is a constant, and P_{jo} is the permutation operator. In this case, we can write out an expansion for $L_{jo}(\lambda)$

$$L_{jo}(\lambda)=P_{jo}\left(1+H_{jo}(\lambda-\eta)+\frac{1}{2!}B_{jo}(\lambda-\eta)^2+\frac{1}{3!}C_{jo}(\lambda-\eta)^3+\ldots\right). \tag{61}$$

Substituting this into (59), we obtain the expansion of $\ln\tau_N(\lambda)$ in powers of $\lambda-\eta$:

$$\ln\tau_N(\lambda)=\ln\tau_N(\eta)+H(\lambda-\eta)+\frac{1}{2!}J(\lambda-\eta)^2+\frac{1}{3!}K(\lambda-\eta)^3+\ldots \tag{62}$$

with

$$H=\sum_j H_{j,j-1} \tag{63}$$

$$J=\sum_j(B_{j,j-1}-H^2_{j,j-1})+\sum_j[H_{j+1,j},H_{j,j-1}] \tag{64}$$

and

$$\begin{aligned}K={}&\sum_j(C_{j,j-1}+2H^3_{j,j-1})-\tfrac{3}{2}\sum_j(H_{j,j-1}B_{j,j-1}+B_{j,j-1}H_{j,j-1})+\tfrac{3}{2}\sum_j[H_{j+1,j},B_{j,j-1}-H^2_{j,j-1}]\\&+\tfrac{3}{2}\sum_j[B_{j+1,j}-H^2_{j+1,j},H_{j,j-1}]+\tfrac{1}{2}\sum_j[H_{j+1,j},[H_{j+1,j},H_{j,j-1}]]\\&+\tfrac{1}{2}\sum_j[[H_{j+1,j},H_{j,j-1}],H_{j,j-1}]+2\sum_j[[H_{j+1,j},H_{j,j-1}],H_{j-1,j-2}].\end{aligned} \tag{65}$$

Applying the procedure outlined above to the Heisenberg XXZ model in a magnetic field, we have

$$\begin{aligned}\ln\tau_N(\lambda)={}&\ln\tau_N(\eta)-\frac{1}{J\sin 2\eta}\left(H-\frac{NW}{2}\right)(\lambda-\eta)\\&-\frac{1}{2!}\frac{1}{J^2\sin^2 2\eta}\left(J(-\mathrm{i})j+\frac{(W+V)^2}{4}\sum_{j=1}^N\sigma_j^z+NJ^2\right)(\lambda-\eta)^2\\&+\frac{1}{3!}\frac{1}{J^3\sin^3 2\eta}\left[k-\left(2J^2+\frac{V^2}{2}\right)\left(H-\frac{NW}{2}\right)\right.\\&\left.+\frac{W+V}{2}\left(2J^2+V^2-\frac{(W+V)^2}{2}\right)\sum_{j=1}^N\sigma_j^z+\frac{NV^3}{8}\right](\lambda-\eta)^3+\ldots\end{aligned} \tag{66}$$

with

$$(-\mathrm{i})j = J\sum_{j=1}^{N}(\sigma_{j+1}^{+}\sigma_{j}^{z}\sigma_{j-1}^{-}-\sigma_{j+1}^{-}\sigma_{j}^{z}\sigma_{j-1}^{+})$$
$$+\frac{V}{2}\sum_{j=1}^{N}[\sigma_{j+1}^{z}(\sigma_{j}^{+}\sigma_{j-1}^{-}-\sigma_{j}^{-}\sigma_{j-1}^{+})+(\sigma_{j+1}^{+}\sigma_{j}^{-}-\sigma_{j+1}^{-}\sigma_{j}^{+})\sigma_{j-1}^{z}] \tag{67}$$

and

$$k = 2J^{3}\sum_{j=1}^{N}(\sigma_{j+1}^{+}\sigma_{j}^{z}\sigma_{j-1}^{z}\sigma_{j-2}^{-}+\sigma_{j+1}^{-}\sigma_{j}^{z}\sigma_{j-1}^{z}\sigma_{j-2}^{+})$$
$$+J^{2}V\sum_{j=1}^{N}[\sigma_{j+1}^{+}\sigma_{j-1}^{-}+\sigma_{j+1}^{-}\sigma_{j-1}^{+}+\tfrac{1}{2}(\sigma_{j+1}^{z}-\sigma_{j}^{z})\sigma_{j-1}^{z}$$
$$+\sigma_{j+1}^{z}(\sigma_{j}^{+}\sigma_{j-1}^{z}\sigma_{j-2}^{-}+\sigma_{j}^{-}\sigma_{j-1}^{z}\sigma_{j-2}^{+})$$
$$+(\sigma_{j+1}^{+}\sigma_{j}^{z}\sigma_{j+1}^{-}+\sigma_{j+1}^{-}\sigma_{j}^{z}\sigma_{j-1}^{+})\sigma_{j-2}^{z}$$
$$-2(\sigma_{j+1}^{+}\sigma_{j}^{-}-\sigma_{j+1}^{-}\sigma_{j}^{+})(\sigma_{j-1}^{+}\sigma_{j-2}^{-}-\sigma_{j-1}^{-}\sigma_{j-2}^{+})]$$
$$+\frac{JV^{2}}{2}\sum_{j=1}^{N}\sigma_{j+1}^{z}(\sigma_{j}^{+}\sigma_{j-1}^{-}+\sigma_{j}^{-}\sigma_{j-1}^{+})\sigma_{j-2}^{z}+\frac{V^{3}}{8}\sum_{j=1}^{N}\sigma_{j}^{z}\sigma_{j-1}^{z}. \tag{68}$$

Similar results can also be obtained for the coupled spin model (31):

$$\ln\tau_{N}(\lambda)=\ln\tau_{N}\left(\frac{\pi}{4}\right)-H\left(\lambda-\frac{\pi}{4}\right)-\frac{1}{2!}\left((-\mathrm{i})j+\frac{\mu^{2}}{4}\sum_{j=1}^{N}(\sigma_{j}^{z}+\tau_{j}^{z})+2N\right)\left(\lambda-\frac{\pi}{4}\right)^{2}$$
$$+\frac{1}{3!}\left[k-(2+\tfrac{5}{8}U^{2})H+\mu\left(1+\frac{\mu^{2}}{4}\right)\sum_{j=1}^{N}(\sigma_{j}^{z}+\tau_{j}^{z})\right]\left(\lambda-\frac{\pi}{4}\right)^{3}+\ldots \tag{69}$$

with

$$(-\mathrm{i})j=\sum_{j=1}^{N}\left(\sigma_{j+1}^{+}\sigma_{j}^{z}\sigma_{j-1}^{-}-\sigma_{j+1}^{-}\sigma_{j}^{z}\sigma_{j-1}^{+}+\frac{U}{2}(\sigma_{j}^{+}\sigma_{j-1}^{-}-\sigma_{j}^{-}\sigma_{j-1}^{+})(\tau_{j}^{z}+\tau_{j-1}^{z})+(\sigma\leftrightarrow\tau)\right) \tag{70}$$

and

$$k=2\sum_{j=1}^{N}[\sigma_{j+1}^{+}\sigma_{j}^{z}\sigma_{j-1}^{z}\sigma_{j-2}^{-}+\sigma_{j+1}^{-}\sigma_{j}^{z}\sigma_{j-1}^{z}\sigma_{j-2}^{+}+(\sigma\to\tau)]$$
$$-U\sum_{j=1}^{N}[2(\sigma_{j+1}^{+}\sigma_{j}^{-}-\sigma_{j+1}^{-}\sigma_{j}^{+})(\tau_{j}^{+}\tau_{j-1}^{-}-\tau_{j}^{-}\tau_{j-1}^{+})$$
$$+(\sigma_{j}^{+}\sigma_{j-1}^{-}-\sigma_{j}^{-}\sigma_{j-1}^{+})(\tau_{j}^{+}\tau_{j-1}^{-}-\tau_{j}^{-}\tau_{j-1}^{+})-(\sigma_{j+1}^{+}\sigma_{j}^{z}\sigma_{j-1}^{-}+\sigma_{j+1}^{-}\sigma_{j}^{z}\sigma_{j-1}^{+})$$
$$\times(\tau_{j+1}^{z}+\tau_{j}^{z}+\tau_{j-1}^{z})-\tfrac{1}{2}(\sigma_{j+1}^{z}\tau_{j}^{z}+\sigma_{j}^{z}\tau_{j}^{z}+\sigma_{j}^{z}\sigma_{j+1}^{z})+(\sigma\leftrightarrow\tau)]$$
$$+\frac{U^{2}}{2}\sum_{j=1}^{N}[(\sigma_{j}^{+}\sigma_{j-1}^{-}+\sigma_{j}^{-}\sigma_{j-1}^{+})\tau_{j}^{z}\tau_{j-1}^{z}+(\sigma\leftrightarrow\tau)]+\frac{U^{3}}{8}\sum_{j=1}^{N}\sigma_{j}^{z}\tau_{j}^{z}. \tag{71}$$

Transforming back to the original fermion operators, we obtain the first two conserved currents for a 1D small-polaron model

$$\begin{aligned}(-\mathrm{i})\mathscr{j}=&-J\sum_{j=1}^{N}(a_{j+1}^{\dagger}a_{j-1}-a_{j-1}^{\dagger}a_{j+1})\\&+V\sum_{j=1}^{N}[n_{j+1}(a_{j}^{\dagger}a_{j-1}-a_{j-1}^{\dagger}a_{j})+(a_{j+1}^{\dagger}a_{j}-a_{j}^{\dagger}a_{j+1})n_{j-1}+a_{j}^{\dagger}a_{j-1}-a_{j-1}^{\dagger}a_{j}]\end{aligned}\tag{72}$$

$$\begin{aligned}\mathscr{k}=&\,2J^{3}\sum_{j=1}^{N}(a_{j+1}^{\dagger}a_{j-2}+a_{j-2}^{\dagger}a_{j+1})\\&-J^{2}V\sum_{j=1}^{N}[2(a_{j+1}^{\dagger}n_{j}a_{j-1}+a_{j-1}^{\dagger}n_{j}a_{j+1})\\&+2(n_{j}-n_{j+1})n_{j-1}+2n_{j+1}(a_{j}^{\dagger}a_{j-2}+a_{j-2}^{\dagger}a_{j})+2(a_{j+1}^{\dagger}a_{j-1}+a_{j-1}^{\dagger}a_{j+1})n_{j-2}\\&+2(a_{j+1}^{\dagger}a_{j}-a_{j}^{\dagger}a_{j+1})(a_{j-1}^{\dagger}a_{j-2}-a_{j-2}^{\dagger}a_{j-1})-3(a_{j+1}^{\dagger}a_{j-1}+a_{j-1}^{\dagger}a_{j+1})]\\&+JV^{2}\sum_{j=1}^{N}[2n_{j+1}(a_{j}^{\dagger}a_{j-1}+a_{j-1}^{\dagger}a_{j})n_{j-2}-n_{j+1}(a_{j}^{\dagger}a_{j-1}+a_{j-1}^{\dagger}a_{j})\\&-(a_{j+1}^{\dagger}a_{j}+a_{j}^{\dagger}a_{j+1})n_{j-1}+\tfrac{1}{2}(a_{j}^{\dagger}a_{j-1}+a_{j-1}^{\dagger}a_{j})]\\&+\frac{V^{2}}{2}\sum_{j=1}^{N}(n_{j}n_{j-1}-n_{j})+\frac{NV^{3}}{8}\end{aligned}\tag{73}$$

and for the 1D Hubbard model

$$\begin{aligned}(-\mathrm{i})\mathscr{j}=&-\sum_{j,s}(a_{j+1s}^{\dagger}a_{j-1s}-a_{j-1s}^{\dagger}a_{j+1s})\\&+U\sum_{j}[(a_{j\uparrow}^{\dagger}a_{j-1\uparrow}-a_{j-1\uparrow}^{\dagger}a_{j\uparrow})(n_{j\downarrow}+n_{j-1\downarrow})+(a_{j\downarrow}^{\dagger}a_{j-1\downarrow}-a_{j-1\downarrow}^{\dagger}a_{j\downarrow})(n_{j\uparrow}+n_{j-1\uparrow})]\\&-U\sum_{j,s}(a_{js}^{\dagger}a_{j-1s}-a_{j-1s}^{\dagger}a_{js})\end{aligned}\tag{74}$$

$$\begin{aligned}\mathscr{k}=&\,2\sum_{j,s}(a_{j+1s}^{\dagger}a_{j-2s}+a_{j-2s}^{\dagger}a_{j+1s})\\&-2U\sum_{j}[(a_{j+1\uparrow}^{\dagger}a_{j\uparrow}-a_{j\uparrow}^{\dagger}a_{j+1\uparrow})(a_{j\downarrow}^{\dagger}a_{j-1\downarrow}-a_{j-1\downarrow}^{\dagger}a_{j\downarrow})\\&+(a_{j\uparrow}^{\dagger}a_{j-1\uparrow}-a_{j-1\uparrow}^{\dagger}a_{j\uparrow})(a_{j+1\downarrow}^{\dagger}a_{j\downarrow}-a_{j\downarrow}^{\dagger}a_{j+1\downarrow})\\&+(a_{j\uparrow}^{\dagger}a_{j-1\uparrow}-a_{j-1\uparrow}^{\dagger}a_{j\uparrow})(a_{j\downarrow}^{\dagger}a_{j-1\downarrow}-a_{j-1\downarrow}^{\dagger}a_{j\downarrow})\\&+(a_{j+1\uparrow}^{\dagger}a_{j-1\uparrow}+a_{j-1\uparrow}^{\dagger}a_{j+1\uparrow})(n_{j+1\downarrow}+n_{j\downarrow}+n_{j-1\downarrow})\\&+(a_{j+1\downarrow}^{\dagger}a_{j-1\downarrow}+a_{j-1\downarrow}^{\dagger}a_{j+1\downarrow})(n_{j+1\uparrow}+n_{j\uparrow}+n_{j-1\uparrow})\\&-2(n_{j+1\uparrow}n_{j\downarrow}+n_{j\uparrow}n_{j\downarrow}+n_{j\uparrow}n_{j+1\downarrow})]\\&+3U\sum_{j,s}(a_{j+1s}^{\dagger}a_{j-1s}+a_{j-1s}^{\dagger}a_{j+1s}+2n_{js})\\&+U^{2}\sum_{j}[(a_{j\uparrow}^{\dagger}a_{j-1\uparrow}+a_{j-1\uparrow}^{\dagger}a_{j\uparrow})(2n_{j\downarrow}n_{j-1\downarrow}-n_{j\downarrow}-n_{j-1\downarrow})\\&+(a_{j\downarrow}^{\dagger}a_{j-1\downarrow}+a_{j-1\downarrow}^{\dagger}a_{j\downarrow})(2n_{j\uparrow}n_{j-1\uparrow}-n_{j\uparrow}-n_{j-1\uparrow})]\\&+\frac{U^{2}}{2}\sum_{j,s}(a_{js}^{\dagger}a_{j-1s}+a_{j-1s}^{\dagger}a_{js})\\&+\frac{U^{3}}{4}\sum_{j}(2n_{j\uparrow}n_{j\downarrow}-n_{j\uparrow}-n_{j\downarrow})+\frac{NU}{2}\left(3+\frac{U^{2}}{4}\right).\end{aligned}\tag{75}$$

5. Conclusion

In this paper we have presented two different forms of the Lax pairs for both a 1D small-polaron model and the 1D Hubbard model. Our results show that not all forms of the Lax pairs are physically reasonable when we attempt to tackle the problems in the framework of QISM. Here we wish to stress that we have not succeeded in finding the solution to the Yang–Baxter relations for a special 2D statistical model generated by the local monodromy matrix (44) because of the enormity of the calculation. In our opinion, however, this monodromy matrix will yield a class of solutions to the Yang–Baxter relations provided

$$a_+a_- + b_+b_- - c^2 = 0 \qquad \frac{a_+b_+}{a_-b_-} = \Gamma. \tag{76}$$

Here Γ is a constant independent of the spectral parameter λ. We will return to this problem in a future publication, along with a comparison of our results with those of Shastry [14].

Also, we have derived an expansion of the transfer matrix $T(\lambda)$ through third order in $\lambda - \eta$. This makes it possible to explicitly construct the first two non-trivial conserved currents for both models. We think our results may provide a basis for finding the boost operator for the Hubbard model [15, 16]. As was noted by Itoyama and Thacker [17], the boost operator plays a central role in the construction of lattice Virasoro algebra in the Baxter eight-vertex model.

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